The Fundamental Principles of Composite Material Stiffness Predictions

David Richardson
Contents

• Description of example material for analysis
• Prediction of Stiffness using…
  – Rule of Mixtures (ROM)
  – ROM with Efficiency Factor
  – Hart Smith 10% rule
  – Classical Laminate Analysis
    • Simplified approach
• Overview of misconceptions in material property comparison between isotropic materials and composites
Lamina Axis Notation

Diagram taken from Harris (1999)
Example Material for Analysis

• M21/35%/UD268/T700
  – A common Aerospace uni-directional pre-preg material called HexPly M21 from Hexcel
• $E_f = 235$ GPa $E_m = 3.5$ GPa
• $\rho_f = 1.78$ g/cm$^3$ $\rho_m = 1.28$ g/cm$^3$
• $W_r = 35\%$ (composite resin weight fraction)
• Layup = (0/0/0/+45/-45/0/0/0/0)
Stage 1

• Convert fibre weight fraction of composite to fibre volume fraction
  – Fibre weight fraction used by material suppliers
  – Fibre volume fraction needed for calculations
Fibre Volume Fraction

- Fibre mass fraction of M21 = 65% (0.65)
  - Data sheet says material is 35% resin by weight, therefore 65% fibre by weight

- Calculation of fibre volume fraction

- The resulting volume fraction is 57.2%

\[ V_f = \frac{W_f}{\rho_f} = \frac{W_f}{\rho_f} + \frac{W_m}{\rho_m} \]
Methods of Stiffness Prediction

- Rule of Mixtures (with efficiency factor)
- Hart-Smith 10% Rule
  - Used in aerospace industry as a quick method of estimating stiffness
- Empirical Formulae
  - Based solely on test data
- Classical Laminate Analysis
  - LAP software
Rule of Mixtures

• A composite is a mixture or combination of two (or more) materials
• The Rule of Mixtures formula can be used to calculate / predict…
  – Young’s Modulus (E)
  – Density
  – Poisson’s ratio
  – Strength (UTS)
    • very optimistic prediction
    • 50% usually measured in test
    • Strength very difficult to predict – numerous reasons
Rule of Mixtures for Stiffness

• Rule of Mixtures for Young’s Modulus
• Assumes uni-directional fibres
• Predicts Young’s Modulus in fibre direction only

\[ E_c = E_f V_f + E_m V_m \]

- \[ E_c = 235 \times 0.572 + 3.5 \times 0.428 \]
- \[ E_c = 136 \text{ GPa} \]
Rule of Mixtures: Efficiency Factor

• The Efficiency Factor or Krenchel factor can be used to predict the effect of fibre orientation on stiffness
• This is a term that is used to factor the Rule of Mixtures formula according to the fibre angle
  – See following slide
Reinforcing Efficiency

\[ \eta_\theta = \sum a_n \cos^4 \theta \]

\( a_n \) = proportion of total fibre content
\( \theta \) = angle of fibres
\( \eta_\theta \) = composite efficiency factor (Krenchel)

\[ E_c = \eta_\theta E_f V_f + E_m V_m \]
Efficiency (Krenchel) Factor

Diagram taken from Harris (1999)
Prediction of $E$ for Example Ply

\[ E_c = \eta_\theta E_f V_f + E_m V_m \]

$E_f = 235$ GPa
$E_m = 3.5$ GPa
$V_f = 0.572$

\[ E(\theta) = (\cos^4 \theta \times 235 \times 0.572) + (3.5 \times 0.428) \]

Predicted modulus versus angle plotted on following slide
Prediction of Tensile Modulus (Efficiency Factor)

Tensile Modulus (GPa) vs. Angle (degrees)
Efficiency Factor for Laminate

- Layup = (0/0/0/+45/-45/0/0/0)
- $\eta = \cos^4\theta$
  - $0^\circ = \eta = 1$
  - $45^\circ = \eta = 0.25$
  - $90^\circ = \eta = 0$

- Laminate in X-direction
  - $(6/8 \times 1) + (2/8 \times 0.25)$
  - $(0.75 + 0.0625)$
  - $0.8125$

- Laminate in Y-direction
  - $(6/8 \times 0) + (2/8 \times 0.25)$
  - $(0 + 0.0625)$
  - $0.0625$
Prediction of $E$ for Example Ply

$$E_c = \eta \theta E_f V_f + E_m V_m$$

$E_f = 235$ GPa $E_m = 3.5$ GPa $V_f = 0.572$

$E_x = (0.8125 \times 235 \times 0.572) + (3.5 \times 0.428)$

$E_x = 109 + 1.5 = 110.5$ GPa

$E_y = (0.0625 \times 235 \times 0.572) + (3.5 \times 0.428)$

$E_y = 8.4 + 1.5 = 9.9$ GPa
Ten-Percent Rule

- Hart-Smith 1993
  - Each 45° or 90° ply is considered to contribute one tenth of the strength or stiffness of a 0° ply to the overall performance of the laminate
  - Rapid and reasonably accurate estimate
  - Used in Aerospace industry where standard layup [0/±45/90] is usually used

\[
E_x = E_{11} \cdot (0.1 + 0.9 \times \% \text{ plies at } 0°)
\]

\[
\sigma_x = \sigma_{11} \cdot (0.1 + 0.9 \times \% \text{ plies at } 0°)
\]

\[
G_{xy} = E_{11} \cdot (0.028 + 0.234 \times \% \text{ plies at } \pm 45°)
\]
Calculation of $E_{11}$ for Ply

- Using Rule of Mixtures

$$E_{11} = E_f V_f + E_m V_m$$

$$E_{11} = 235 \times 0.572 + 3.5 \times 0.428$$

- $E_{11} = 136$ GPa

- Layup = (0/0/0/+45/-45/0/0/0)

- 6/8 = 75% of plies in zero degree direction
Ten-Percent Rule

\begin{itemize}
  \item \( E_x = E_{11} \times (0.1 + 0.9 \times \% \text{ plies at } 0^\circ) \)
  \item \( E_x = 136 \times (0.1 + (0.9 \times 0.75)) \)
  \item \( E_x = 136 \times (0.775) \)
  \item \( E_x = 105.4 \text{ GPa} \)

  \item \( E_y = E_{11} \times (0.1 + 0.9 \times \% \text{ plies at } 0^\circ) \)
  \item \( E_y = 136 \times (0.1 + (0.9 \times 0)) \)
  \item \( E_y = 136 \times (0.1) \)
  \item \( E_y = 13.6 \text{ GPa} \)
\end{itemize}
Classical Laminate Analysis

- 4 elastic constants are needed to characterise the in-plane macroscopic elastic properties of a ply
  - $E_{11} =$ Longitudinal Stiffness
  - $E_{22} =$ Transverse Stiffness
  - $\nu_{12} =$ Major Poisson’s Ratio
  - $G_{12} =$ In-Plane Shear Modulus
Elastic Constant Equations

• \( E_{11} = \) Longitudinal Stiffness (Rule of Mixtures Formulae)

\[
E_c = E_f V_f + E_m (1 - V_f)
\]

• \( E_{22} = \) Transverse Stiffness (Inverse Rule of Mixtures Formulae (Reuss Model))

\[
\frac{1}{E_c} = \frac{V_f}{E_f} + \frac{V_m}{E_m}
\]

• \( \nu_{12} = \) Major Poisson’s Ratio (Rule of Mixtures for Poisson’s Ratio)

\[
\nu_{12} = \nu_f V_f + \nu_m (1 - V_f)
\]

• \( G_{12} = \) In-Plane Shear Modulus (Inverse Rule of Mixtures for Shear)

\[
\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{(1 - V_f)}{G_m}
\]
Calculation of $E_{11}$ for Ply

• Using Rule of Mixtures

\[ E_{11} = E_f V_f + E_m V_m \]

• $E_{11} = 235 \times 0.572 + 3.5 \times 0.428$

• $E_{11} = 136 \text{ GPa}$
Calculation of $E_{22}$ for Ply

- Using Inverse Rule of Mixtures Formulae (Reuss Model)
- $E_{22} = 8$ GPa

\[
\frac{1}{E_c} = \frac{V_f}{E_f} + \frac{V_m}{E_m}
\]
Calculate Poisson’s Ratio ($v_{12}$) for Ply

- Using Rule of Mixtures formula
- However, we do not know
  - Poisson’s ratio for carbon fibre
  - Poisson’s ratio for epoxy matrix
  - We would need to find these for accurate prediction
- We will assume a Poisson’s Ratio ($v$) of 0.3
Calculate Shear Modulus \((G_{12})\) of Ply

- Using Inverse Rule of Mixtures formula
- \(G\) for carbon fibre = 52 GPa (from test)
- \(G\) for epoxy = 2.26 GPa (from test)
  - Both calculated using standard shear modulus formula \(G = E/(2(1+v))\)
- \(G_{12}\) for composite = 5 GPa

\[
\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{(1 - V_f)}{G_m}
\]
Resulting Properties of Ply

- $E_{11} = 136$ GPa
- $E_{22} = 8$ GPa
- $\nu_{12} = 0.3$
- $G_{12} = 5$ GPa
Matrix Representation

• 4 material elastic properties are needed to characterise the in-plane behaviour of the linear elastic orthotropic ply
  – We conveniently define these in terms of measured engineering constants (as above)
  – These are usually expressed in matrix form
    • due to large equations produced
    • and subsequent manipulations required

• The stiffness matrix \([Q]\)
• The compliance matrix \([S]\) (inverse of stiffness)
Off-axis Orientation & Analysis

• The stiffness matrix is defined in terms of principal material directions, $E_{11}$, $E_{22}$
• However, we need to analyse or predict the material properties in other directions
  – As it is unlikely to be loaded only in principal direction
• We use stress transformation equations for this
  – Related to Mohr’s stress circle
• The transformation equations are written in matrix form
  – They have nothing to do with the material properties, they are merely a rotation of stresses.
Single Ply

- [6 x 6] stiffness matrix [C] or
- [6 x 6] compliance matrix [S]
  - Often reduced stiffness matrix [Q] for orthotropic laminates [3 x 3]
  - Orthotropic = 3 mutually orthogonal planes of symmetry
  - 4 elastic constants characterise the behaviour of the laminate
    - $E_1$, $E_2$, $\nu_{12}$, $G_{12}$
Stiffness & Compliance Matrices

### Stiffness Matrix $[Q]$

Calculates laminate stresses from laminate strains

\[
\begin{bmatrix}
\sigma_x \\ \sigma_y \\ \tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{21} & Q_{22} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}
\end{bmatrix}.
\]

### Compliance Matrix $[S]$

Calculates laminate strains from laminate stresses (inverse of compliance)

\[
\begin{bmatrix}
\varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & S_{16} \\
S_{21} & S_{22} & S_{26} \\
S_{16} & S_{26} & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\ \sigma_y \\ \tau_{xy}
\end{bmatrix}.
\]
The stress transformation equation that relates known stresses in the z, y coordinate system to stresses in the L, T coordinate system. These are related to the transformation performed using Mohr’s stress circle.

\[
\begin{pmatrix}
\sigma_L \\
\sigma_T \\
\tau_{LT}
\end{pmatrix} =
\begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & 2 \sin \theta \cos \theta \\
\sin^2 \theta & \cos^2 \theta & -2 \sin \theta \cos \theta \\
-\sin \theta \cos \theta & \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta
\end{bmatrix}
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{pmatrix}.
\]
Transformed Stiffness Components

\[
\bar{Q}_{11} = Q_{11} \cos^4 \theta + Q_{22} \sin^4 \theta + (2Q_{12} + 4Q_{66}) \cos^2 \theta \sin^2 \theta
\]

\[
\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \cos^2 \theta \sin^2 \theta + Q_{12} (\cos^4 \theta + \sin^4 \theta)
\]

\[
\bar{Q}_{22} = Q_{11} \sin^4 \theta + Q_{22} \cos^4 \theta + (2Q_{12} + 4Q_{66}) \cos^2 \theta \sin^2 \theta
\]

\[
\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \cos^2 \theta \sin^2 \theta + Q_{66} (\cos^4 \theta + \sin^4 \theta)
\]

\[
\bar{Q}_{16} = (Q_{11} - 2Q_{66} - Q_{12}) \cos^3 \theta \sin \theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos \theta \sin^3 \theta
\]

\[
\bar{Q}_{26} = (Q_{11} - 2Q_{66} - Q_{12}) \cos \theta \sin^3 \theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos^3 \theta \sin \theta
\]
Transformed Compliance Components

\[ S_{11} = S_{11} \cos^4 \theta + S_{22} \sin^4 \theta + (2S_{12} + S_{66}) \cos^2 \theta \sin^2 \theta \]

\[ S_{12} = (S_{11} + S_{22} - S_{66}) \cos^2 \theta \sin^2 \theta + S_{12}(\cos^4 \theta + \sin^4 \theta) \]

\[ S_{22} = S_{11} \sin^4 \theta + S_{22} \cos^4 \theta + (2S_{12} + S_{66}) \cos^2 \theta \sin^2 \theta \]

\[ S_{66} = 4(S_{11} + S_{22} - 2S_{12}) \cos^2 \theta \sin^2 \theta + S_{66}(\cos^2 \theta - \sin^2 \theta)^2 \]

\[ S_{16} = (2S_{11} - S_{66} - 2S_{12}) \cos^3 \theta \sin \theta - (2S_{22} - 2S_{12} - S_{66}) \cos \theta \sin^3 \theta \]

\[ S_{26} = (2S_{11} - S_{66} - 2S_{12}) \cos \theta \sin^3 \theta - (2S_{22} - 2S_{12} - S_{66}) \cos^3 \theta \sin \theta \]
Individual Compliance & Stiffness terms

\[
S_{11} = \frac{1}{E_L}, \quad S_{12} = -\frac{\mu_{LT}}{E_L} = -\frac{\mu_{TL}}{E_T}, \quad S_{22} = \frac{1}{E_T}, \quad S_{66} = \frac{1}{G_{LT}}.
\]

\[
Q_{11} = \frac{E_L}{(1 - \mu_{LT} \mu_{TL})}, \quad Q_{12} = \frac{\mu_{LT} E_T}{(1 - \mu_{LT} \mu_{TL})} = \frac{\mu_{TL} E_L}{(1 - \mu_{LT} \mu_{TL})},
\]

\[
Q_{22} = \frac{E_T}{(1 - \mu_{LT} \mu_{TL})}, \quad \text{and} \quad Q_{66} = G_{LT}.
\]
CLA Derived Formula

• Formula from
  – Engineering Design with Polymers and Composites, J.C.Gerdeen et al.

• Suitable for calculating Young’s Modulus of a uni-directional ply at different angles.

\[
\frac{1}{E_{xx}} = \frac{\cos^4 \theta}{E_{11}} + \frac{\sin^4 \theta}{E_{22}} + \left( \frac{1}{G_{12}} - \frac{2v_{12}}{E_{11}} \right) \cos^2 \theta \sin^2 \theta
\]
Example Material ($\theta = 0^\circ$)

- $E_{11} = 136$ GPa, $E_{22} = 8$ GPa
- $\nu_{12} = 0.3$, $G_{12} = 5$ GPa
- $1/E_x = 1/136 + 0 + 0$
- $E_x$ for $0^\circ$ fibres = 136 GPa

$$\frac{1}{E_x} = \frac{\cos^4 \theta}{E_{11}} + \frac{\sin^4 \theta}{E_{22}} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right) \cos^2 \theta \sin^2 \theta$$
Example Material ($\theta = 45^\circ$)

- $E_{11} = 136$ GPa, $E_{22} = 8$ GPa
- $\nu_{12} = 0.3$, $G_{12} = 5$ GPa
- $1/E_x = 0.25/136 + 0.25/8 + (1/5-0.6/136) \times 0.25$
- $E_x$ for $45^\circ$ fibres = 12.2 GPa
Example Material (θ = 90°)

- $E_{11} = 136$ GPa, $E_{22} = 8$ GPa
- $\nu_{12} = 0.3$, $G_{12} = 5$ GPa
- $1/E_x = 0 + 1/8 + 0$
- $E_x$ for 90° fibres = 8 GPa

\[
\frac{1}{E_x} = \frac{\cos^4 \theta}{E_{11}} + \frac{\sin^4 \theta}{E_{22}} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right)\cos^2 \theta \sin^2 \theta
\]
Properties of Laminate

• CFRE (0/0/0/+45/-45/0/0/0)
  • 0° = 136 GPa
  • 45° = 12.2 GPa
  • 90° = 8 GPa

• Laminate in X-direction
  • (6/8 × 136) + (2/8 × 12.2)
  • 102 + 3 = 105 GPa

• Laminate in Y-direction
  • (6/8 × 8) + (2/8 × 12.2)
  • 6 + 3 = 9 GPa

• Note: This simplified calculation ignores the effect of coupling between extension and shear
Classical Laminate Analysis

• The simplified laminate analysis approach taken ignores the effect of coupling between extension and shear

• Classical Laminate Analysis takes full account of this effect
  – However, this is too long and complex for hand calculations
  – Therefore build a spreadsheet or use software such as LAP (Laminate Analysis Programme)
Summary of Results

For HexPly M21 Pre-preg with (0/0/0/+45/-45/0/0/0) layup

<table>
<thead>
<tr>
<th></th>
<th>ROM &amp; Efficiency factor</th>
<th>Hart-Smith 10% Rule</th>
<th>ROM of CLA rotation formula</th>
<th>Classical Laminate Analysis LAP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x$</td>
<td>110.5</td>
<td>105.4</td>
<td>105.0</td>
<td>107.1</td>
</tr>
<tr>
<td>$E_y$</td>
<td>9.9</td>
<td>13.6</td>
<td>9.0</td>
<td>15.3</td>
</tr>
</tbody>
</table>
Prediction of Tensile Modulus

![Graph showing prediction of tensile modulus vs. angle (degrees).]

- CLA
- ROM
- Hart-Smith
Procedure for Structural Analysis

- Classical Laminate Analysis to provide
  - Relationship between in-plane load and strain
  - Relationship between out of plane bending moments and curvatures
    - Plate properties from the A, B and D matrices
- Then analysis is equivalent to classical analysis of anisotropic materials
  - FE analysis
  - Standard formulas to check stiffness
Material Property Comparison

• Young’s Modulus of materials
  – Isotropic Materials
    • Aluminium = 70 GPa
    • Steel = 210 GPa
    • Polymers = 3 GPa
  – Fibres
    • Carbon = 240 GPa
      – common high strength T700 fibre
    • Glass = 70 GPa
Care with Property Comparison

- $E$ for Aluminium $= 70$ GPa
- $E$ for Glass fibre $= 70$ GPa
- However….
  - 50% fibre volume fraction
    - $E$ now $35$ GPa in $x$-direction
    - $E$ in $y$-direction $=$ matrix $= 3$ GPa
  - 0/90 woven fabric
    - 50% of material in each direction
    - $E$ now $17.5$ GPa in $x$ and $y$ directions
    - $E$ at $45^\circ = 9$ GPa
- Glass reinforced polymer composite now has a low stiffness compared to Aluminium!
Elasticity until yield point, then followed by large range of plasticity. Design to yield - therefore beneficial plasticity safety zone.

**UD CFRE** – 50% Fv (E = 120 GPa)
CFRE is elastic until ultimate failure (no plasticity)

**0/90 CFRE** – 50% Fv (E = 60 GPa)
Elasticity until failure (no plasticity)

**Quasi-Isotropic CFRE** – 50% Fv (E = 45 GPa)
Elasticity until failure (no plasticity)

**Aluminium** (E = 70 GPa)
Elastic until yield point, then followed by large range of plasticity. Design to yield - therefore beneficial plasticity safety zone.
References & Bibliography

• Engineering Design with Polymers and Composites
  • J.C.Gerdeen, H.W.Lord & R.A.L.Rorrer
  • Taylor and Francis, 2006

• Engineering Composite Materials
  • Bryan Harris, Bath University

• Composite Materials - UWE E-learning resource
  • David Richardson, John Burns & Aerocomp Ltd.
Contact Details

- Dr David Richardson
- Room 1N22
- Faculty of Engineering and Technology
- University of the West of England
- Frenchay Campus
- Coldharbour Lane
- Bristol
- BS16 1QY
- Tel: 0117 328 2223
- Email: David4.Richardson@uwe.ac.uk